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Imbedding Theorems for Spaces of Hypersingular Integrals and Bessel Potentials

WALTER TREBELS

Lehrstuhl A für Mathematik, Technological University of Aachen, Aachen, Germany

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1. INTRODUCTION AND NOTATION

The main purpose of this paper is to give an approximation-theoretic approach to some imbedding problems treated in Euclidean n space by Stein [10] and Wheeden [15]. In particular, we are interested in equivalent norms for the space of Bessel potentials L_α^p for all p, $1 \leqslant p \leqslant \infty$, and all $\alpha > 0$, in terms of hypersingular integrals; these results will be derived in a unified way.

Observing that L_α^p is the Favard class of a large set of radial approximation processes, we will show that the equivalence of the L_α^p norm with the norm involving appropriate hypersingular integrals can be interpreted as a saturation problem. Actually, the hypersingular integral in question (smoothened in some sense) can be rewritten as an approximation process on f minus f multiplied by the optimal or saturation order of this process [see formulas (2.6) and (2.7)]. By verifying the hypothesis of a general saturation theorem (see Lemma 2.2) in this particular case, we first arrive at an equivalence relation with the L_α^p norm which, however, is not of the desired form. But by using some elementary estimates of Salem–Zygmund type [8] we can sharpen this result and obtain one implying Stein's [10] as well as parts of those of Wheeden [15].

An application of the Marcinkiewicz–Mikhlin multiplier theorem will establish a connection between coordinatewise and radial hypersingular integrals in the reflexive case $1 < p < \infty$ (see also [13]).

Furthermore, if in the same case $1 < p < \infty$ one inserts in the hypersingular integral a homogeneous function of degree zero, integrable on the unit sphere, the imbedding of this modified hypersingular integral into L_α^p corresponds to a multiplier problem in Fourier transform. This will be solved only for $\alpha > [n/2] + 1$ by applying the Marcinkiewicz–Mikhlin multiplier theorem in its version due to Hörmander [6]. The case $O < \alpha \leqslant [n/2] + 1$ could be treated by tedious computations along the lines of Wheeden's [15] calculations, and will be omitted.

Here we are only interested in imbedding theorems for the space L_α^p, and therefore we restrict ourselves to norm statements; pointwise analogs may be obtained by properly modifying Wheeden's [15] proof.

Before proceeding further we list some conventions and notation to be used. Let N be the set of positive integers, $x = (x_1, \dots, x_n)$ a point in Euclidean n space E_n, e^k the point x with all $x_j = 0$ except for x_k which is 1, $j = (j_1, \dots, j_n)$ an n-tuple of nonnegative integers. We set $x \cdot y = \sum_{k=1}^n x_k y_k$, $|x|^2 = x \cdot x$, $x' = x/|x|$ $(|x| > 0)$. Σ is the unit sphere $|x| = 1$, $x^j = x_1^{j_1} \dots x_n^{j_n}$, and $|j| = j_1 + \cdots + j_n$. Constants will be denoted by C; one or more subscripts may indicate quantities on which it depends. $[\alpha]$ denotes the largest integer less than or equal to α. $L^p(E_n)$ is the space of (Lebesgue-) measurable functions f for which the norm $\|f\|_p$ is finite. Here

$$\|f\|_p = \left\{\int_{E_n} |f(x)|^p \, dx\right\}^{1/p}, \quad 1 \leqslant p < \infty, \quad \|f\|_\infty = \operatorname*{ess\,sup}_{x \in E_n} |f(x)|.$$

M is the set of bounded measures μ normed by $\|d\mu\|_1 = \int |d\mu|$ (integrals without limits are taken over all of E_n), C is the space of uniformly continuous bounded functions f, with $\|f\|_C = \sup_x |f(x)|$. Defining the convolution of $\mu \in M$ and $f \in L^p$ by

$$f * d\mu(x) = (2\pi)^{-n/2} \int f(x - y) \, d\mu(y),$$

we have $\|f * d\mu\|_p \leqslant \|f\|_p \|d\mu\|_1$ for $1 \leqslant p \leqslant \infty$.

The s-th difference $(s \in N)$ of a measurable function f and the s-th central difference of f, with increment $u \in E_n$, are given by

$$\Delta_u^s f(x) = \sum_{m=0}^{s} (-1)^m \binom{s}{m} f(x + (s - m)u), \qquad \bar{\Delta}_u^s f(x) = \Delta_u^s f\left(x - \frac{s}{2} u\right),$$

respectively. Finally, the Fourier–Stieltjes transform of $\mu \in M$ and the Fourier transform of $f \in L^1$ are defined by

$$[d\mu]^\wedge(v) = (2\pi)^{-n/2} \int e^{-iv \cdot x} d\mu(x), \qquad f^\wedge(v) = (2\pi)^{-n/2} \int e^{-iv \cdot x} f(x) \, dx,$$

respectively.

Let $\alpha > 0$; then

$$G_\alpha(x) = [2^{(\alpha-2)/2} \Gamma(\alpha/2)]^{-1} |x|^{(\alpha-n)/2} K_{(n-\alpha)/2}(|x|)$$

is called the Bessel kernel of order α; here ($\zeta \geqslant 0$),

$$K_\beta(\zeta) = \frac{\pi}{2}\frac{I_{-\beta}(\zeta) - I_\beta(\zeta)}{\sin \beta\pi}, \qquad I_\beta(\zeta) = \sum_{m=0}^{\infty} \frac{(\zeta/2)^{\beta+2m}}{m!\,\Gamma(\beta+m+1)}$$

are the modified Bessel functions of order β of the third and first kind, respectively.

G_α is a nonnegative, integrable function [7, p. 341] with $\int G_\alpha(x)\,dx = (2\pi)^{n/2}$; its Fourier transform is given by

$$G_\alpha\hat{\ }(v) = (1 + |v|^2)^{-\alpha/2}.$$

With the aid of the n-dimensional Bessel kernel G_α, the space of radial Bessel potentials is defined by ($\alpha > 0$)

$$L_\alpha{}^p = \left\{ f \in L^p; f = G_\alpha * \begin{cases} d\mu, \ \mu \in M, \text{ if } & p = 1 \\ h, \ h \in L^p, \text{ if } & 1 < p \leqslant \infty \end{cases} \right\}. \tag{1.1}$$

As usual, $L_\alpha{}^p$ is normed by

$$\|f\|_{1,\alpha} = \|d\mu\|_1 \ (p = 1), \quad \|f\|_{p,\alpha} = \|h\|_p \ (1 < p \leqslant \infty).$$

This (radial) space $L_\alpha{}^p$ will be compared, in case $1 < p < \infty$, with ($\gamma_k > 0$)

$$L^p_{(\gamma_1,\ldots,\gamma_n)} = \{ f \in L^p; f = ({}_1G_{\gamma_k} * h_k)_{(k)}\,, \qquad h_k \in L^p, \ \ 1 < p \leqslant \infty, \ \ k = 1,\ldots, n\}. \tag{1.2}$$

(For the sake of simplicity we omit the case $p = 1$.) Here ${}_1G_\alpha$ is the one-dimensional Bessel kernel, and

$$({}_1G_\alpha * g)_{(k)}(x) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} g(x - \eta e^k)\,{}_1G_\alpha(\eta)\,d\eta$$

is the convolution in the k-th coordinate. $L^p_{(\gamma_1,\ldots,\gamma_n)}$ is normed by

$$\|f\|_{p,(\gamma_1,\ldots,\gamma_n)} = \sum_{k=1}^{n} \|h_k\|_p\,.$$

2. Hypersingular Integrals on L^p, $1 \leqslant p \leqslant \infty$

In this section we introduce the hypersingular integral

$$\int |y|^{-n-\alpha} \Delta_y^{2s} f(x)\,dy, \tag{2.1}$$

equip it with a suitable norm, and show that this norm is equivalent to the L_α^p norm for all $\alpha > 0$ and all p, $1 \leqslant p \leqslant \infty$. Actually we shall prove the following

THEOREM 2.1. *The following norms on L_α^p, are equivalent for $n = 1, 2,\ldots$, $\alpha > 0$:*

(a) *In case $1 \leqslant p \leqslant \infty$:*

(i) $\|f\|_{p,\alpha}$,

(ii) $\|f\|_p + \sup_{\tau>0} \| \int | y |^{-n-\alpha} \bar{\Delta}_y^{2s} K_\tau * f \, dy \|_p \quad (0 < \alpha < 2s, s \in N)$,

where the smoothening kernel K_τ, a linear combination of Weierstrass *kernels, is given by*

$$K_\tau(u) = - \sum_{m=1}^{2s} (-1)^m \binom{2s}{m} (2\tau m^2)^{-n/2} \exp\{- | u |^2/4\tau m^2\},$$

(iii) $\|f\|_p + \sup_{\epsilon>0} \| \int_{|y|\geqslant\epsilon} | y |^{-n-\alpha} \bar{\Delta}_y^{2s} f \, dy \|_p$.

(b) *In case $1 < p < \infty$,* (iii) *converges as $\epsilon \to 0^+$, and we have as another equivalent norm*

(iii)* $\|f\|_p + \| \int | y |^{-n-\alpha} \bar{\Delta}_y^{2s} f \, dy \|_p$.

This remains true for $p = 1$, provided the measure $\mu \in M$ in (1.1) *is absolutely continuous.*

Remark. Since the kernel $| y |^{-n-\alpha}$ and the domains of integration are radial, we may replace the $(2s)$-th central difference $\bar{\Delta}_y^{2s} f$ in (iii) and (iii)* by

$$\sum_{m=0}^{s} (-1)^k \binom{2s}{m} f(x + (s - m)y) - \frac{1}{2} (-1)^s \binom{2s}{s} f(x).$$

In this form for $s = 1$, $0 < \alpha < 2$ and $1 < p < \infty$, the equivalence of (i) and (iii)* is already in Stein [10]. Wheeden [15] proves Stein's theorem for $1 \leqslant p \leqslant \infty$ [in the modified version of (iii) if $p = \infty$] and can replace $| y |^{-n-\alpha}$ by $\Omega(y') \, | y |^{-n-\alpha}$ in case $1 < p < \infty$ (This will be discussed in Section 3). He also states an equivalence relation in case $\alpha \geqslant 2$, whereby his hypersingular integral involves as regularization process a Taylor expansion of f whereas a higher difference of f will be used by us. The advantage of our approach is that we need not distinguish between various α intervals and different p values. With the help of these general results we then obtain more specific ones depending on further properties, such as the reflexivity of L^p $(1 < p < \infty)$ or, in case $p = 1$, the absolute continuity of μ.

Proof of Theorem 2.1(a). First we show the equivalence of (i) and (ii) by applying a general saturation theorem. To this end, we observe that

$$I_\tau(x) = \int |y|^{-n-\alpha} \bar{\Delta}_y^{2s} K_\tau * f(x)\, dy \tag{2.2}$$

is absolutely convergent for all x ($\tau > 0$ fixed) and all $f \in L^p$, $1 \leqslant p \leqslant \infty$. Indeed, split up $I_\tau(x)$ into two terms

$$\Big(\int_{|y|\leqslant 1} + \int_{|y|\geqslant 1}\Big) |y|^{-n-\alpha} \bar{\Delta}_y^{2s} K_\tau * f(x)\, dy = I_\tau^1(x) + I_\tau^2(x).$$

Obviously, I_τ^2 converges absolutely, since $K_\tau \in L^{p'}$ and thus

$$|\bar{\Delta}_y^{2s} K_\tau * f(x)| \leqslant \sum_{m=0}^{2s} \binom{2s}{m} \| K_\tau \|_{p'} \| f \|_p$$

by Hölder's inequality. Concerning I_τ^1, let us expand $\bar{\Delta}_y^{2s} K_\tau$ into a Taylor series at the point $y = 0$. On account of the identity

$$\sum_{m=0}^{2s} (-1)^m \binom{2s}{m} (s-m)^l = 0 \qquad (0 \leqslant l \leqslant 2s-1), \tag{2.3}$$

all derivatives of $\bar{\Delta}_y^{2s} K_\tau$ of order less than $2s$ vanish at $y = 0$. Therefore,

$$\bar{\Delta}_y^{2s} K_\tau = C_s \sum_{m=0}^{2s} (-1)^m \binom{2s}{m}$$

$$\times \int_0^1 (1-\eta)^{2s-1} \sum_j y^j D_y{}^j K_\tau(x + (s-m)\, y\eta)\, d\eta, \tag{2.4}$$

where the last summation is extended over all permutations of j with $|j| = 2s$. Observing that $|y^j| \leqslant |y|^{2s}$ and that $D^j K_\tau$ belongs to all $L^{p'}$ spaces, $1 \leqslant p' \leqslant \infty$, we can show as before that I_τ^1, too, converges absolutely for fixed $\tau > 0$.

To show the equivalence of (i) and (ii) we now need the following saturation result (see [14]):

LEMMA 2.2. *Let* $1 \leqslant p \leqslant \infty$; *let* $f \in C$ *in case* $p = \infty$, *and let* $f \in L^p$ *in case* $1 \leqslant p < \infty$. *Define, for* $\rho > 0$

$$I_\rho(f; x) = (2\pi)^{-n/2} \int f(x-y)\, d\nu(\rho y), \qquad \nu \in M, \qquad \int d\nu = (2\pi)^{n/2}.$$

If ν satisfies the condition

there exist constants $\alpha > 0$, $c \neq 0$ and a measure $\lambda \in M$ with $\int d\lambda = (2\pi)^{n/2}$ such that (2.5)

$$[d\nu]^\wedge(v) - 1 = c[d\lambda]^\wedge(v) \mid v \mid^\alpha \qquad (v \in E_n),$$

then

$$c_1 \| f \|_{p,\alpha} \leqslant \| f \|_p + \sup_{\rho > 0} \| \rho^\alpha \{ I_\rho(f; \cdot) - f \} \|_p \leqslant c_2 \| f \|_{p,\alpha} \tag{2.6}$$

for some constants c_1, $c_2 > 0$, both independent of f.

In order to apply Lemma 2.2, comparing the hypersingular integral in 2.1 (a-ii) with the second expression in (2.6), we observe that I_τ has to be of type $\rho^\alpha \{I_\rho(f; \cdot) - f\}$. Set $\rho = \tau^{-1/2}$ and

$$I_\tau(x) = \tau^{-\alpha/2} \{ f * d\nu_\alpha(\tau^{-1/2} \cdot)(x) - f(x) \}, \tag{2.7}$$

the measure ν_α being given by (A any Lebesgue-measurable set)

$$\nu_\alpha(A) = \int_A \left\{ \int \mid y \mid^{-n-\alpha} \bar{\Delta}_y^{2s} K_1(u)\, dy\, du + d\sigma \right\}, \tag{2.8}$$

where σ is the discrete measure with mass $(2\pi)^{n/2}$ at the origin. Since by the same reasoning as before (K_1, $D^j K_1$ are integrable on E_n)

$$\int \mid y \mid^{-n-\alpha} dy \int \bar{\Delta}_y^{2s} K_1(u)\, du \tag{2.9}$$

is absolutely convergent, it is obvious by Fubini's theorem that $\nu_\alpha \in M$. Also, ν_α is normalized; for σ is normalized and the inner integral in (2.9) vanishes on account of the identity (2.3) for $l = 0$ and the normalization of K_1. It remains to verify condition (2.5): since $[d\sigma]^\wedge(v) = 1$,

$$[d\nu_\alpha]^\wedge(v) - 1 = (2\pi)^{-n/2} \iint \bar{\Delta}_y^{2s} K_1(u) \mid y \mid^{-n-\alpha} dy\, e^{-iv\cdot u}\, du.$$

An interchange of integrations gives

$$[d\nu_\alpha]^\wedge(v) - 1 = K_1^\wedge(v) \int \mid y \mid^{-n-\alpha} (e^{(i/2)v\cdot y} - e^{-(i/2)v\cdot y})^{2s}\, dy. \tag{2.10}$$

If $n \geqslant 2$, we can subject the latter integral to an orthogonal transformation (cf. [2, p. 70]) and obtain

$$[d\nu_\alpha]^\wedge(v) - 1 = \int \mid y \mid^{-n-\alpha} (2i \sin(y_1/2))^{2s}\, dy\, K_1^\wedge(v) \mid v \mid^\alpha.$$

Since this relation is achieved for $n = 1$ by a simple substitution, condition (2.5) is satisfied and hence (i) and (ii) of Theorem 2.1 are equivalent.

In case $n \geqslant 2$, we could replace $\bar{\Delta}_y^{2s}$ by Δ_y^l ($l \in N$, $0 < \alpha < l$); but then we would have to distinguish between the cases $n = 1$ and $n \geqslant 2$.

It remains to show the equivalence with (iii). Obviously, by Fatou's lemma,

$$\| I_\tau \|_p \leqslant \liminf_{\epsilon \to 0+} \left\| \int_{|y| \geqslant \epsilon} | y |^{-n-\alpha} \bar{\Delta}_y^{2s} K_\tau * f(x)\, dy \right\|_p$$

$$\leqslant \liminf_{\epsilon \to 0+} \| K_\tau \|_1 \left\| \int_{|y| \geqslant \epsilon} | y |^{-n-\alpha} \bar{\Delta}_y^{2s} f\, dy \right\|_p, \tag{2.11}$$

and since $\| K_\tau \|_1 \leqslant \sum_{m=1}^{2s} \binom{2s}{m} (2\pi)^{n/2}$, we have estimated (ii) by (iii).

To prove the converse, we proceed analogously, relying on Butzer–Görlich [3] and Sunouchi [11]. We have

$$\left\| \int_{|y| \geqslant \sqrt{\tau}} | y |^{-n-\alpha} \bar{\Delta}_y^{2s} f\, dy \right\|_p \leqslant \left\| \int_{|y| \geqslant \sqrt{\tau}} | y |^{-n-\alpha} \bar{\Delta}_y^{2s} \{K_\tau * f - f\}\, dy \right\|_p$$

$$+ \left\| \int_{|y| \leqslant \sqrt{\tau}} | y |^{-n-\alpha} \bar{\Delta}_y^{2s} K_\tau * f \right\|_p + \| I_\tau \|_p \equiv I_1 + I_2 + I_3 \tag{2.12}$$

so that only I_1 and I_2 have to be estimated by (ii) or, equivalently, by (i). To this end, we need two approximation-theoretical arguments.

LEMMA 2.3. *If $f \in L_\alpha^p$, for $0 < \alpha < 2s$, $s \in N$, then*

$$\| \bar{\Delta}_y^{2s} \{K_\tau * f - f\} \|_p \leqslant C_{\alpha,s}\, \tau^{\alpha/2} \| f \|_{p,\alpha}\,; \tag{2.13}$$

$$\| \bar{\Delta}_y^{2s} K_\tau * f \|_p \leqslant C_{\alpha,s}^*\, \tau^{(\alpha-2s)/2} \,| y |^{2s} \| f \|_{p,\alpha}\,. \tag{2.14}$$

In view of this lemma, the proof of part (a) of Theorem 2.1 is now complete since, by the generalized Minkowski inequality and formula (2.13),

$$I_1 \leqslant C_{\alpha,s}\, \tau^{\alpha/2} \| f \|_{p,\alpha} \int_{|y| \geqslant \sqrt{\tau}} | y |^{-n-\alpha} dy = C_{\alpha,s,n} \| f \|_{p,\alpha}\,,$$

and, analogously, by formula (2.14),

$$I_2 \leqslant C_{\alpha,s}^*\, \tau^{(\alpha-2s)/2} \| f \|_{p,\alpha} \int_{|y| \leqslant \sqrt{\tau}} | y |^{2s-\alpha-n}\, dy = C_{\alpha,s,n} \| f \|_{p,\alpha}\,.$$

Proof of Lemma 2.3. By a straightforward calculation,

$$K_\tau * f(x) - f(x) = -(4\pi\tau)^{-n/2} \int \Delta_y^{2s} f(x) \exp\{- | y |^2/4\tau\}\, dy$$

and hence, since $f \in L_\alpha^p$ implies

$$\| \Delta_y^l f \|_p \leqslant C \mid y \mid^\alpha \| f \|_{p,\alpha} \qquad (0 < \alpha < l) \tag{2.15}$$

(see, e.g., [14]), relation (2.13) follows by the generalized Minkowski inequality. Concerning (2.14), we remark that as a special case of a theorem of Butzer–Scherer [4,5] one has

$$\| D^j K_\tau * f \|_p \leqslant C_{\alpha,s}\, \tau^{(\alpha-2s)/2} \| f \|_{p,\alpha} \qquad \text{for} \quad | j | = 2s. \tag{2.16}$$

Hence, a Taylor expansion of $\bar{\Delta}_y^{2s} K_\tau * f$ at $y = 0$ gives, analogously to (2.4),

$$\bar{\Delta}_y^{2s} K_\tau * f(x) = C_s \sum_{m=0}^{2s} (-1)^m \binom{2s}{m}$$
$$\times \int_0^1 (1-\eta)^{2s-1} \sum_j y^j D_y{}^j K_\tau * f(x + (s-m)\, y\eta)\, d\eta.$$

The relations (2.16) and $| y^j | \leqslant | y |^{2s}$ yield the desired inequality (2.14) with the aid of the generalized Minkowski inequality.

Now we prove (b) of Theorem 2.1 by an application of the Banach–Steinhaus theorem. L_α^p and L^p are Banach spaces with respect to the norms $\| f \|_{p,\alpha} = \| h \|_p$ [where $\mu(A) = \int_A h(x)\, dx$ in case $p = 1$] and $\| f \|_p$, respectively; the operators $\int_{|y| \geqslant \epsilon} | y |^{-n-\alpha} \ldots$ on L_α^p into L^p, $1 \leqslant p < \infty$, are uniformly bounded by part (a); if $f \in L_\beta^p$, $\beta > \alpha$, then we have strong convergence because [cf. (2.15)]

$$\left\| \int_{\epsilon_1 \leqslant |y| \leqslant \epsilon_2} | y |^{-n-\alpha} \bar{\Delta}_y^{2s} f\, dy \right\|_p \leqslant C_{\beta,s} \| f \|_{p,\beta} \int_{\epsilon_1 \leqslant |y| \leqslant \epsilon_2} | y |^{\beta-\alpha-n}\, dy$$

which tends to zero as $\epsilon_2 \to 0$; thus, on account of the completeness of the L^p spaces, there clearly exists $g \in L^p$ such that

$$g = s\text{-}\lim_{\epsilon \to 0} \int_{|y| \geqslant \epsilon} | y |^{-n-\alpha}\, \bar{\Delta}_y^{2s} f\, dy.$$

Since L_β^p is dense in L_α^p, $1 \leqslant p < \infty$, the Banach–Steinhaus theorem assures convergence for all $f \in L_\alpha^p$. Since $\lim_{\epsilon \to 0} \| \ \|_p \leqslant \sup_{\epsilon > 0} \| \ \|_p$ and (2.11) holds, the equivalence of (i) and (iii)* follows.

Remark 2.4. A result analogous to Theorem 2.1(a), in the periodic, one-dimensional case, was proved by Sunouchi [12], applying a counterpart of Lemma 2.2. He shows that

$$\int_\epsilon^\infty y^{-1-\alpha}\, \bar{\Delta}_y^{2s} f\, dy = C\, \epsilon^{-\alpha} \{ f * k_\epsilon - f \} \tag{2.17}$$

and verifies condition (2.5) for the kernel k_ϵ.

3. The Reflexive Case $1 < p < \infty$

In this concluding section we shall obtain more detailed results in the case $1 < p < \infty$. To this end we first compare the above radial hypersingular integral with a so-called coordinatewise one, i.e., an integral of the form

$$\int_\epsilon^\infty \eta^{-1-\alpha}\, \bar{\Delta}^{2s}_{\eta e^k} f(x)\, d\eta.$$

We have (for $1 < p \leqslant 2$ see [13])

Theorem 3.1. *The following expressions are norms on* L_α^p, $1 < p < \infty$, $\alpha > 0$, *equivalent to those of* (i)–(iii)* *in Theorem* 2.1:

(iv) $\|f\|_{p,(\alpha,\ldots,\alpha)}$,

(v) $\displaystyle \|f\|_p + \sum_{k=1}^n \sup_{\epsilon>0} \left\| \int_\epsilon^\infty \eta^{-1-\alpha}\, \bar{\Delta}^{2s}_{\eta e^k} f\, d\eta \right\|_p$.

Proof. The equivalence of (i) and (iv) follows readily from Nikolskii's book [7, p. 74]; he has shown, using the Marcinkiewicz–Mikhlin multiplier theorem, that

$$(1 + v_k^2)^{\alpha/2}\,(1 + |\, v\, |^2)^{-\alpha/2} \quad (k = 1, \ldots, n),$$

$$(1 + |\, v\, |^2)^{\alpha/2} \left(\sum_{k=1}^n (1 + v_k^2)^{\alpha/2}\right)^{-1}$$

are multipliers of type (L^p, L^p), $1 < p < \infty$. Thus, it remains only to establish the equivalence of (iv) and (v). This can be done with the help of the above multiplier theorem by comparing norm (ii) of Theorem 2.1 with a corresponding coordinatewise one for each coordinate, and then using the same estimates of Salem–Zygmund type as before.

However, we prefer a proof depending on a saturation theorem for singular integrals with product kernels and n parameters $r = (r_1, \ldots, r_n)$:

$$I_r(f; x) = (2\pi)^{-n/2} \int f(x - y) \prod_{k=1}^n d\nu_k(r_k y_k), \tag{3.1}$$

where

$$\nu_k \in M(E_1) \quad \text{and} \quad \int_{-\infty}^\infty d\nu_k = \sqrt{(2\pi)}.$$

For this approximation process, the following saturation result holds:

LEMMA 3.2. *Let $1 \leqslant p \leqslant \infty$; let $f \in C$ in case $p = \infty$ and let $f \in L^p$ in case $1 \leqslant p < \infty$. If the one-dimensional measures ν_k of formula* (3.1) *satisfy the condition*

$$\text{there exist constants } \gamma_k > 0,\ c_k \neq 0 \text{ and measures } \lambda_k \in M(E_1) \text{ normalized to satisfy } \int_{-\infty}^{\infty} d\lambda_k = \sqrt{(2\pi)} \text{ such that, for } k = 1,\dots, n, \tag{3.2}$$

$$[d\nu\ ]^{\wedge}(\zeta) - 1 = c_k[d\lambda_k]^{\wedge}(\zeta) \mid \zeta \mid^{\gamma_k} \qquad \text{for all } \zeta \in E_1,$$

then the following norms are equivalent:

(i) $\| f \|_{p,(\gamma_1,\dots,\gamma_n)}$,

(ii) $\| f \|_p + \sum_{k=1}^{n} \sup_{r_k > 0} \| r_k^{\gamma_k} \{ I_{r_k}(f; \cdot) - f \} \|_p$,

where

$$I_{r_k}(f; x) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} f(x - \eta e^k)\, d\nu_k(r_k \eta).$$

As was shown by Berens–Nessel [1], such saturation theorems involving singular integrals with decomposable product kernels are essentially one dimensional because the condition

$$\| I_r(f; \cdot) - f \|_p = O\left(\sum_{k=1}^{n} r_k^{-\gamma_k} \right)$$

is equivalent to the following set of conditions:

$$\| I_{r_k}(f; \cdot) - f \|_p = O(r_k^{-\gamma_k}) \qquad (k = 1,\dots, n).$$

Thus, with slight modifications, the proof of Lemma 2.2 in case $n = 1$ gives the desired result for each $I_{r_k}(f; \cdot)$.

Now—as pointed out in Remark 2.4—Sunouchi [12] has already proved the representation (2.17) and condition (3.2), so that the equivalence of (iv) and (v) is obvious if we take $\gamma_k = \alpha$, $k = 1,\dots, n$.

Now let us see what happens if we replace the factor $\mid y \mid^{-n-\alpha}$ in (2.1) by $\Omega(y') \mid y \mid^{-n-\alpha}$, where Ω is a homogeneous function of degree zero, integrable on the unit sphere. First we consider

$$\int \Omega(y') \mid y \mid^{-n-\alpha} \bar{\Delta}_y^{2s} K_\tau * f(x)\, dy \qquad (0 < \alpha < 2s). \tag{3.3}$$

As before, this hypersingular integral converges absolutely for fixed $\tau > 0$.

For sufficiently smooth functions f (e.g., infinitely differentiable and rapidly decreasing) we can take its Fourier transform, obtaining

$$K_\tau\hat{\ }(v)\, f\hat{\ }(v) \int \Omega(y') \mid y \mid^{-n-\alpha} (e^{(i/2)v\cdot y} - e^{-(i/2)v\cdot y})^{2s}\, dy. \tag{3.4}$$

On the other hand, for such smooth functions f, we have as the Fourier transform of $I_\tau(x)$—calculated as in the proof of Theorem 2.1—

$$K_\tau\hat{\ }(v)\, f\hat{\ }(v) \mid v \mid^{\alpha} \int \mid y \mid^{-n-\alpha} (e^{(i/2)y_1} - e^{-(i/2)y_1})^{2s}\, dy. \tag{3.5}$$

Comparison of formulas (3.4) and (3.5) suggests the interpretation [by inserting $\mid v \mid^{\alpha} \mid v \mid^{-\alpha}$ in (3.4)] of

$$\psi(v) = \mid v \mid^{-\alpha} \int \Omega(y') \mid y \mid^{-n-\alpha} (e^{(i/2)v\cdot y} - e^{-(i/2)v\cdot y})^{2s}\, dy \tag{3.6}$$

as a multiplier of type (L^p, L^p). Now $\psi(v)$ is a multiplier of type (L^2, L^2) for all $\alpha > 0$ because ($v' \in \Sigma$, the unit sphere)

$$\begin{aligned} \mid \psi(v) \mid &= \left| \int \Omega(y') \mid y \mid^{-n-\alpha} (e^{(i/2)v'\cdot y} - e^{-(i/2)v'\cdot y})^{2s}\, dy \right| \\ &\leqslant \int_{|y|\leqslant 1} \mid \Omega(y') \mid \mid y \mid^{2s-n-\alpha}\, dy + 2^{2s} \int_{|y|\geqslant 1} \mid \Omega(y') \mid \mid y \mid^{-n-\alpha}\, dy \end{aligned}$$

which is clearly finite since $\Omega \in L^1(\Sigma)$ and $0 < \alpha < 2s$.

Is $\psi(v)$ also a multiplier for the other p values, $1 < p < \infty$? To study this question we use the Marcinkiewicz–Mikhlin multiplier theorem in its improved form due to Hörmander [6]:

Let $\psi \in L^\infty$ and assume that

$$\sup_{R>0} R^{-n} \int_{\frac{1}{2}R\leqslant|v|\leqslant 2R} \mid R^{|j|} D^j \psi(v) \mid^2 dv < \infty \quad (0 \leqslant \mid j \mid \leqslant [n/2] + 1). \tag{3.7}$$

Then ψ is a multiplier of type (L^p, L^p), $1 < p < \infty$.

Let j' and j'' be n-tuples of nonnegative integers. By Leibniz's formula we obtain

$$\begin{aligned} D^j\psi(v) &= \sum_{j'+j''=j} D^{j'} \mid v \mid^{-\alpha} D^{j''} \int \Omega(y') \mid y \mid^{-n-\alpha} (e^{(i/2)v\cdot y} - e^{-(i/2)v\cdot y})^{2s}\, dy \\ &= \sum (D^{j'} \mid v \mid^{-\alpha}) \int \Omega(y') \mid y \mid^{-n-\alpha} \\ &\quad \times \sum_{m=0}^{2s} (-1)^m \binom{2s}{m} (i(s-m)y)^{j''} e^{i(s-m)v\cdot y}\, dy. \end{aligned}$$

For $|j''| \leqslant |j| \leqslant [n/2] + 1 < \alpha$, the latter integral is absolutely convergent (v fixed); indeed, the product of the factor $(e^{(i/2)v\cdot y} - e^{-(i/2)v\cdot y})^{2s-|j''|}$ with $y^{j''} \cdot |y|^{-n-\alpha}$ is integrable in the neighborhood of the origin, whereas the choice of α guarantees the convergence at infinity. For $0 < \alpha \leqslant [n/2] + 1$, the behavior at infinity causes difficulties and we do not consider this case any further.

Now substituting $|v| y = u$ and multiplying $D^j\psi(v)$ by $|v|^{|j|}$, we see that there exists a constant $C_{\alpha,n,s,\Omega} = C$ such that $|v|^{|j|} |D^j\psi(v)| \leqslant C$ for all j, $0 \leqslant |j| \leqslant [n/2] + 1$; hence, in particular, the condition (3.7) is satisfied. Applying Lemma 2.3, we obtain by the same reasoning as in the proof of Theorem 2.1(a) the following

THEOREM 3.3. *Let $f \in L_\alpha{}^p$, $1 < p < \infty$, and let $\alpha > [n/2] + 1$. If $\Omega \in L^1(\Sigma)$ is homogeneous of degree zero, then*

$$\sup_{\epsilon>0} \left\| \int_{|y|\geqslant\epsilon} \Omega(y') \, |y|^{-n-\alpha} \, \Delta_y^{2s} f \, dy \right\|_p \leqslant \|f\|_{p,\alpha} \qquad (2s > \alpha).$$

This holds for all $\alpha > 0$, if $p = 2$.

Remark. Arguing as in the proof of Theorem 2.1(b), we see that this hypersingular integral converges as $\epsilon \to 0$ and that the above estimate remains valid for the limit. Wheeden [15, II] has shown this relation for $s = 1$ and $0 < \alpha < 2$; his idea of proof consists in applying the Marcinkiewicz interpolation theorem directly to the operator $\int_{|y|\geqslant\epsilon} \Omega(y') \, |y|^{-n-\alpha} \ldots$ by establishing the L^2 boundedness of this transformation and the following weak type (1.1) condition:

$$m \left\{x; \left| \int_{|y|\geqslant\epsilon} \Omega(y') \, |y|^{-n-\alpha} \{f(x-y) - f(x)\} \, dy \right| > \zeta > 0 \right\}$$
$$\leqslant \text{const } \zeta^{-1} \|f\|_{1,\alpha} .$$

For the other α values he states in [15, I] some analogous results [instead of a $(2s)$-th difference of f he uses a corresponding Taylor expansion of f] under the condition that Ω is infinitely differentiable. Following this proof expanding Ω into spherical harmonics one can show by pure computation that ψ is a multiplier of type (L^p, L^p), $1 < p < \infty$, if Ω satisfies the conditions of the Calderón–Zygmund theory on singular integrals (see, e.g., [6]).

We conclude with the observation that if a concrete example of Ω is given for which one can compute explicitly its Fourier transform, then it should be quite easy to show that $\psi(v)$ of (3.6) is a multiplier, e.g., if sgn v_k, $v_k/|v|$ or arbitrary products of these are considered.

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